TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 280, Number 1, November 1983

TOEPLITZ OPERATORS ON BOUNDED SYMMETRIC DOMAINS

BY
HARALD UPMEIER

ABSTRACT. In this paper Jordan algebraic methods are applied to study Toeplitz operators on the Hardy space $H^2(S)$ associated with the Shilov boundary S of a bounded symmetric domain D in $\mathbf{C}^n$ of arbitrary rank. The Jordan triple system $Z \approx \mathbf{C}^n$ associated with D is used to determine the relationship between Toeplitz operators and differential operators. Further, it is shown that each Jordan triple idempotent $e \in Z$ induces an irreducible representation ("e-symbol") of the C^*-algebra $\mathfrak{T}$ generated by all Toeplitz operators T_f with continuous symbol function f.

0. Introduction. Toeplitz operators on the boundary $\mathbf{T} = \partial\Delta$ of the open unit disc $\Delta \subset \mathbf{C}$ play an important role in function theory of one complex variable (cf. [**9**, Chapter 7]). In several dimensions Toeplitz operators have been mainly studied for *strictly pseudo-convex domains* $D \subset \mathbf{C}^n$ [**4**, **13**, **14**, **24**, **28**], in particular for the Hilbert ball [**7**], using the relationship with pseudo-differential operators. Another class of domains in $\mathbf{C}^n$ generalizing the unit disc is the class of *bounded symmetric domains* (Cartan domains and exceptional domains), which have a more complicated boundary structure compared to the strictly pseudo-convex case.

The aim of this paper is the study of Toeplitz operators T_f with *continuous* symbol function $f \in \mathcal{C}(S)$ on the Shilov boundary S of a bounded symmetric domain D of arbitrary rank r. In the special cases of the Hilbert ball ($r = 1$) and the Lie ball ($r = 2$), the structure of the operators T_f and of the *Toeplitz C^*-algebra*

$$\mathfrak{T} := C^*\big(T_f : f \in \mathcal{C}(S)\big)$$

generated by these operators is well understood (cf. [**1**, **2**, **7**]). The general analysis presented here is based on the *Jordan theoretic characterization* of bounded symmetric domains. By [**16**, **20**] every bounded symmetric domain D can be realized as the *open unit ball* of a uniquely determined *Jordan triple system* $Z \approx \mathbf{C}^n$ for the so-called spectral norm, and the domains of tube type correspond exactly to the Jordan triple systems defined by complexified formally-real *Jordan algebras*. The holomorphic and boundary structure of D can be described algebraically in terms of the associated Jordan structures (cf. [**20**, §6]).

In [**27**] the Jordan triple system corresponding to D has been used to determine an explicit *Peter-Weyl decomposition* of the Hardy space $H^2(S)$ associated with S. Based on the construction of the *conical polynomials* belonging to the irreducible components of $H^2(S)$, our first main result (Theorem 2.1) concerns the relationship

Received by the editors August 25, 1982.
1980 *Mathematics Subject Classification*. Primary 47B35; Secondary 32M15.

between the invariant *differential* and *integral scalar products* on these components. As an application it is shown that, similar to the strictly pseudo-convex case, certain "basic" Toeplitz operators can be described by differential operators defined via the Jordan triple product (Theorem 2.11). In §1 the fundamental kernel functions (Szegö and Poisson kernel) are characterized in Jordan theoretic terms.

Applying the above results, it is shown in §3 that each *tripotent* $e \in Z$ (satisfying the Jordan triple identity $\{ee^*e\} = e$) gives rise to an *irreducible representation* ("e-symbol") of the Toeplitz C^*-algebra $\mathfrak{T}$ on a suitable Hardy space. Actually it can be shown that every irreducible representation of $\mathfrak{T}$ has this form, which implies that $\mathfrak{T}$ is a *solvable* C^*-algebra (in the sense of [**10**]) of length $r = \operatorname{rank}(D)$.

1. Toeplitz operators. Let D be a bounded symmetric domain in a complex vector space Z of finite dimension. Without loss of generality we may assume D is circular and contains the origin. The identity component G of the biholomorphic automorphism group $\operatorname{Aut}(D)$ acts on the Shilov boundary S of D in a natural way and the compact linear group

$$K := \{g \in G: g(0) = 0\}$$

is transitive on S [**20**, Theorem 5.3]. Denote by $L^2(S)$ the Lebesgue space associated with the unique K-invariant probability measure μ on S.

Since D is circular, the *Hardy space* $H^2(S)$ (cf. [**17**, §4]) can be identified with the closure (in $L^2(S)$) of the algebra $\mathfrak{P}(Z)$ of all polynomials on Z. The orthogonal projection $\pi\colon L^2(S) \to H^2(S)$ is called the *Szegö projection*, being induced by the Szegö kernel $\mathbb{S}\colon D \times (D \cup S) \to \mathbf{C}$ (cf. [**17**, §4]). Given $f \in L^\infty(S)$, the bounded operator T_f on $H^2(S)$, defined by

$$T_f h := \pi(fh)$$

for all $h \in H^2(S)$, is called the *Toeplitz operator* with *symbol function* f. Putting $\bar{f}(s) := \overline{f(s)}$ for all $s \in S$, the definitions imply

$$T_f^* = T_{\bar{f}} \tag{1.1}$$

and

$$T_f T_g = T_{fg} \tag{1.2}$$

whenever $g \in H^\infty(S)$ (the bounded holomorphic functions) and $f \in L^\infty(S)$. Here $*$ denotes the Hilbert space adjoint.

Our study of Toeplitz operators will be based on the *Jordan theoretic description of bounded symmetric domains* in terms of Jordan algebras and Jordan triple systems [**16**, **20**]. To indicate this relationship, recall that the Lie algebra $\mathfrak{g}$ of G can be identified with the set of all *complete holomorphic vector fields* on D [**22**, Chapter 9]. Holomorphic vector fields will be regarded as differential operators $A = h(z)(\partial/\partial z)$, where $h\colon D \to Z$ is holomorphic and z denotes the "coordinate" of Z, acting on holomorphic functions $f\colon D \to \mathbf{C}$ via

$$(A \cdot f)(z) := f'(z)h(z),$$

$f'(z)$ denoting the derivative of f in $z \in D$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition with respect to $0 \in D$. By [**20**, Theorem 4.1] there exists a unique *Jordan triple product*

$$Z \times Z \times Z \ni (u, v, w) \mapsto \{uv^*w\} \in Z$$

such that

$$\mathfrak{p} = \{(v - \{zv^*z\})(\partial/\partial z) : v \in Z\}.$$

The domain D can be characterized as the *open unit ball* of the Jordan triple system Z for the so-called spectral norm. It is well known [**16**, **20**] that the above construction establishes a 1-1 correspondence between (circular) bounded symmetric domains and the positive definite hermitian Jordan triple systems, called *JB*-triples* in [**27**]. (By [**6**] this correspondence can be generalized to domains in complex Banach spaces, thus motivating our terminology.) The group K, the largest connected group of *Jordan triple automorphisms* (cf. [**20**, §3]) of Z, will sometimes be denoted by $K = \mathrm{Aut}(Z)^0$. Our first lemma follows from [**16**, Chapter IV, Theorem 6.2; **17**, Remark 4.12 and **19**, Theorem 17.3].

LEMMA 1.1. *For irreducible domains D of rank r and dimension d the Szegö kernel is given by*

$$\mathcal{S}(u, v) = N(u, v)^{-d/r},$$

where N denotes the generic norm of the associated JB-triple Z (cf.* [**19**, §16]).

The homogeneous part of bidegree (1, 1) of the "sesquipolynomial" mapping N: $Z \times Z \to \mathbf{C}$ is called the *generic trace* [**19**, §16] and is a K-invariant scalar product on Z, denoted by $(\,|\,)$. By [**17**, p. 342], the *Poisson kernel* $\mathcal{P}$: $D \times S \to \mathbf{R}$ is given by

$$\mathcal{P}(z, v) := |\mathcal{S}(z, v)|^2 / \mathcal{S}(z, z)$$

for all $(z, v) \in D \times S$. Differentiation and Lemma 1.1 yield

LEMMA 1.2. *Suppose D is irreducible and $v \in S$. Then the real analytic mapping $\mathcal{P}_v$: $z \mapsto \mathcal{P}(z, v)$ has the derivative*

$$\mathcal{P}_v'(0)u = (2d/r)\mathrm{Re}(u \,|\, v),$$

where $(\,|\,)$ denotes the generic trace.

The Shilov boundary of (classical) symmetric domains of rank $\leqslant 2$ is (essentially) a sphere of appropriate dimension. In this special case, Toeplitz operators are closely related to *Calderón-Zygmund operators* (pseudo-differential operators of order 0) [**1**, **2**, **7**]. Using harmonic analysis in $H^2(S)$, we will now study relations between Toeplitz operators and certain differential operators for irreducible domains of arbitrary rank.

According to [**25**] the Hilbert space $H^2(S)$, endowed with the K-invariant *integral scalar product*

$$(f|g)_S := \int_S \overline{f(s)}\, g(s)\, d\mu(s),$$

admits a *Peter-Weyl decomposition* (of the dense subspace $\mathcal{P}(Z)$)

$$\mathcal{P}(Z) = \sum_{m \in \mathbf{N}_+^r}^{\oplus} E_m \tag{1.3}$$

into pairwise inequivalent irreducible unitary K-modules E_m associated with the "signatures" $m = (m_1, \ldots, m_r)$, where

$$\mathbf{N}_+^r := \{m = (m_1, \ldots, m_r) \in \mathbf{N}^r \colon m_1 \geqslant \cdots \geqslant m_r \geqslant 0\}.$$

On the other hand, the generic trace on Z associates a constant coefficient differential operator ∂_p with each polynomial $p \in \mathcal{P}(Z)$ such that

$$\partial_l = v(\partial/\partial z),$$

where $l = v^*$ is defined by $v^*(z) := (z \mid v)$ for all v, $z \in Z$. The K-invariant scalar product

$$(p \mid q)_Z := (\partial_p q)(0)$$

on $\mathcal{P}(Z)$ (cf. [**29**, 2.1.4]) is called the *differential scalar product*. The decomposition (1.3) is orthogonal for both scalar products and

$$(p \mid q)_Z = C_m (p \mid q)_S \tag{1.4}$$

for all $p, q \in E_m$, where $C_m > 0$ depends only on the signature $m \in \mathbf{N}_+^r$. Let C: $\mathcal{P}(Z) \to \mathcal{P}(Z)$ denote the (unbounded) operator in $H^2(S)$ defined by

$$C \mid E_m := C_m \cdot \mathrm{id} \tag{1.5}$$

for all $m \in \mathbf{N}_+^r$.

LEMMA 1.3. *Suppose f, p, $q \in \mathcal{P}(Z)$. Then*

$$(T_f p \mid q)_Z = (p \mid \partial_f q)_Z.$$

PROOF. Using multiplicative properties we may assume f is linear, hence $f = u^*$ for some $u \in Z$. Since the subspaces $\mathcal{P}^k(Z)$ of all k-homogeneous polynomials are mutually orthogonal, we may assume further that $q = (v^*)^k$ and $p \in \mathcal{P}^{k-1}(Z)$ for suitable $k \in \mathbf{N}$ and $v \in Z$. In this case,

$$\begin{aligned}(T_f p \mid q)_Z &= k! f(v) p(v) = k!\,(v \mid u) p(v) = k (v \mid u)(k-1)! p(v) \\ &= k (v \mid u)\big(p \mid (v^*)^{k-1}\big)_Z = (p \mid \partial_f q)_Z. \qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY 1.4. *The adjoint (with respect to $H^2(S)$) of the Toeplitz operator T_f for $f \in \mathcal{P}(Z)$ satisfies*

$$T_f^* = C \partial_f C^{-1},$$

where C is the "diagonal" operator in $H^2(S)$ defined in (1.5).

The transformation properties of the Poisson kernel $\mathcal{P}$ of D (cf. [**17**, §4]) imply that for every $g \in G$, with $z := g(0)$, the image measure $g_* \mu$ has the density function $\mathcal{P}_z(v) := \mathcal{P}(z, v)$ for all $v \in S$, i.e. $g_* \mu = \mathcal{P}_z \cdot \mu$. This leads to the following result which will be considerably sharpened in §2.

PROPOSITION 1.5. *Let Z be an irreducible JB^*-triple of rank r and dimension d. Then the operator identity*

$$\{zv^*z\}\frac{\partial}{\partial z} = \left(v\frac{\partial}{\partial z}\right)^* - \frac{d}{r}T_l = CT_lC^{-1} - \frac{d}{r}T_l$$

holds on $\mathcal{P}(Z)$ for every $v \in Z$ and $l := v^*$.

PROOF. Consider the vector field

$$A := (v - \{zv^*z\})(\partial/\partial z) \in \mathfrak{g}$$

and put $g_t := \exp(tA)$ for all $t \in \mathbf{R}$. Then

$$(p \circ g_t | q \circ g_t)_S = \int_S \bar{p}q\, d((g_t)_*\mu) = \int_S \bar{p}q\mathcal{P}_{g_t(0)}\, d\mu = (p | q\mathcal{P}_{g_t(0)})_S$$

for all $p, q \in \mathcal{P}(Z)$. Differentiation and Lemma 1.2 yield

$$(A \cdot p | q)_S + (p | A \cdot q)_S = (p | T_f q)_S,$$

where $f(z) := (2d/r)\mathrm{Re}(v | z)$. Polarization and Corollary 1.4 yield the assertion. Q.E.D.

A bounded symmetric domain D is said to be of *tube type* if D is holomorphically equivalent to a tube domain over a self-dual homogeneous cone. In terms of the associated JB^*-triple Z, this case is characterized by the existence of a *unitary* element $e \in Z$ (satisfying $\{ee^*z\} = z$ for all $z \in Z$) making Z into a *Jordan algebra* with unit e, product $z \circ w := \{ze^*w\}$ and involution $z^* := \{ez^*e\}$. The Jordan algebras obtained in this way (called *JB^*-algebras* in [**27**]) are precisely the complexifications of formally-real Jordan algebras [**5**, Chapter XI].

Toeplitz operators on symmetric domains of tube type have special properties which derive from the existence of the *norm function* $N \in \mathcal{P}(Z)$ of the corresponding JB^*-algebra Z, normalized by $N(e) = 1$ and related to the generic norm of the underlying JB^*-triple via $N(e - z) = N(e, z^*)$ for all $z \in Z$ (cf. [**19**, p. 178, Theorem 16.11]). By [**27**, Lemma 3.8], the norm function satisfies $|N(z)| = 1$ and $(\partial_l N)(z) = \overline{l(z)}N(z)$ for all $z \in S$ and every linear form l on Z. Applying (1.1) and (1.2), it follows that

$$T_N^* T_N = \mathrm{id} \tag{1.6}$$

and

$$T_N^* T_{\partial_l N} = T_l^*. \tag{1.7}$$

According to [**27**, Theorem 2.6], the cokernel of the isometry T_N can be characterized as follows:

PROPOSITION 1.6. *Let Z be an irreducible JB^*-algebra of rank r with norm function N. Denote by $S' := \{z \in S: N(z) = 1\}$ the "reduced Shilov boundary" of the open unit ball $D \subset Z$. Then* $\ker(T_N^*)$ *is the closure of the vector space*

$$\mathcal{H}(Z) = \{p \in \mathcal{P}(Z): \partial_N p = 0\} = \sum_{m \in \mathbf{N}_+^{r-1}}^{\oplus} E_m$$

of all harmonic polynomials on Z. Further, the restriction mapping $\rho\colon \mathcal{P}(Z) \to L^2(S')$, *defined by* $\rho(p) := p \mid S'$, *induces a Hilbert space isomorphism*

$$\overline{\mathcal{H}(Z)} \approx L^2(S'). \tag{1.8}$$

Here we have used the natural identification

$$\mathbf{N}_+^{r-1} = \{(m_1, \ldots, m_r) \in \mathbf{N}_+^r : m_r = 0\}. \tag{1.9}$$

In view of [**27**, Theorem 3.7] and (1.8), there exists a Hilbert space tensor product realization

$$H^2(S) = \overline{\mathcal{H}(Z)} \otimes H^2(\mathbf{T}),$$

where $\mathbf{T} \subset \mathbf{C}$ denotes the 1-torus. Obviously,

$$T_N = \mathrm{id} \otimes T_\xi,$$

where ξ denotes the identity function on $\mathbf{T}$. Similar tensor product realizations hold for other "basic" Toeplitz operators (cf. [**1**, **2**]):

PROPOSITION 1.7. *Let Z be an irreducible JB*-algebra with norm function N. Given* $v \in Z$, *let* $l := v^*$ *be the associated linear form. Then there exist tensor product representations*

$$T_l = A_v^* \otimes \mathrm{id} + B_v^* \otimes T_\xi \tag{1.10}$$

and

$$T_{\partial_l N} = B_v \otimes \mathrm{id} + A_v \otimes T_\xi, \tag{1.11}$$

where A_v *and* B_v *are bounded linear operators on* $\overline{\mathcal{H}(Z)}$.

PROOF. For any $f \in \mathcal{P}(Z)$, T_f^* leaves $\overline{\mathcal{H}(Z)}$ invariant since $[T_f^*, T_N^*] = [T_N, T_f]^* = 0$. Given $q \in \mathcal{H}(Z)$, Theorem 2.6 of [**27**] results in a unique representation

$$(\partial_l N)q = \sum_{k \geq 0} N^k q_k,$$

where $q_k \in \mathcal{H}(Z)$ for all k. Applying (1.6) and (1.7) yields

$$T_l^* q = \sum_{k \geq 1} N^{k-1} q_k \in \overline{\mathcal{H}(Z)}$$

which implies $q_k = 0$ for all $k > 1$. It follows that

$$(\partial_l N)q = B_v q + N A_v q$$

for all $q \in \overline{\mathcal{H}(Z)}$ and certain bounded operators B_v and A_v on $\overline{\mathcal{H}(Z)}$. This implies (1.11). Further, (1.10) follows with (1.6) and (1.7). Q.E.D.

PROPOSITION 1.8. *Let Z be an irreducible JB*-algebra. Suppose* $u, v \in Z$ *and* $l := v^*$. *Then*

(1) $$A_v = T_l^* \mid \overline{\mathcal{H}(Z)}, \qquad B_v^* = T_{\partial_l N}^* \mid \overline{\mathcal{H}(Z)}.$$

(2) $$[A_u, A_v] = 0 = [B_u, B_v], \qquad [A_u, B_v] = [A_v, B_u],$$
$$[A_u, B_v^*] = 0 = [B_u, A_v^*], \qquad [A_u, A_v^*] = [B_v^*, B_u].$$

The following lemma will be used in §2.

LEMMA 1.9. *Let Z be an irreducible JB^*-algebra of rank r and dimension d. Suppose $v \in Z$ and $k \in \mathbf{N}$. Then for every harmonic polynomial $q \in \mathcal{H}(Z)$,*

$$\{zv^*z\}\frac{\partial}{\partial z}(N^k q) = N^k q_0 + \left(k + 1 - \frac{d}{r}\right)N^{k+1}(B_v^* q),$$

where $q_0 \in \mathcal{H}(Z)$.

PROOF. According to Propositions 1.5 and 1.7,

$$h := \{zv^*z\}(\partial/\partial z)(N^k q) = N^k q_0 + N^{k+1} q_1$$

for suitable polynomials q_0, $q_1 \in \mathcal{H}(Z)$. Given $p \in \mathcal{H}(Z)$, it follows from (1.6), Proposition 1.5, (1.10) and (1.11) that

$$\begin{aligned}(q_1|p)_S &= (h|N^{k+1}p)_S = (N^k q|\partial_l(N^{k+1}p))_S - \frac{d}{r}(N^{k+1}(B_v^* q)|N^{k+1}p)_S \\ &= (k+1)(q|p \cdot \partial_l N)_S - \frac{d}{r}(B_v^* q|p)_S = \left(k + 1 - \frac{d}{r}\right)(B_v^* q|p)_S,\end{aligned}$$

where $l := v^*$. Since p is arbitrary, the assertion follows. Q.E.D.

2. Toeplitz operators and differential operators. In order to make the relation between Toeplitz operators and differential operators indicated in Corollary 1.4 and Proposition 1.5 more precise, we will show how the eigenvalues C_m of the diagonal operator C defined in (1.4) can be computed in a uniform manner from the signature $m = (m_1, \ldots, m_r)$ of the K-module E_m and from certain numerical invariants of the domain D.

Let Z be the JB^*-triple associated with an irreducible bounded symmetric domain D of rank r. An element $e \in Z$ is called a *tripotent* ("triple idempotent") if $\{ee^*e\} = e$. Any *frame* $\{e_1, \ldots, e_r\}$ of minimal orthogonal tripotents [20, §5] induces a *Peirce decomposition*

$$Z = \sum_{0 \leqslant p \leqslant q \leqslant r}^{\oplus} Z_{pq},$$

where $Z_{pq} := \{z \in Z\colon 2\{e_j e_j^* z\} = (\delta_{jp} + \delta_{jq})z \text{ for } 1 \leqslant j \leqslant r\}$ and δ_{jp} denotes the Kronecker symbol. The 3-tuple (r, s, t), defined by $s := \dim Z_{pq}$ $(1 \leqslant p < q \leqslant r)$, $t := \dim Z_{oq}$ $(1 \leqslant q \leqslant r)$ [19, §17], is called the *type* of Z or of D. Then $t = 0$ if and only if Z is a JB^*-algebra.

For $1 \leqslant k \leqslant r$, the JB^*-subtriple

$$Z_k := \sum_{r-k<p\leqslant q \leqslant r}^{\oplus} Z_{pq} \tag{2.1}$$

is a JB^*-algebra with unit element $e_{r+1-k} + \cdots + e_r$. Let $N_k \in \mathcal{P}(Z_k) \subset \mathcal{P}(Z)$ denote the norm function of Z_k (in general, for any subspace $\tilde{Z} \subset Z$, embed $\mathcal{P}(\tilde{Z}) \subset \mathcal{P}(Z)$ via the orthogonal projection onto $\tilde{Z}$). Then it has been shown in [27, Theorem 3.1] (cf. also [15]) that for any signature $m = (m_1, \ldots, m_r) \in \mathbf{N}_+^r$ the polynomial

$$N_m := N_1^{l_1} \cdots N_r^{l_r} \tag{2.2}$$

is the *conical polynomial* (cf. [**29**, 3.3, p. 211]) of E_m, where $l_j := m_j - m_{j+1}$ for $j < r$ and $l_r := m_r$. Define $L := \{g \in K: g(e) = e\}$ for $e := e_1 + \cdots + e_r \in S$. Then

$$P_m := \int_L (N_m \circ g)\, dg \in E_m \tag{2.3}$$

is the unique L-invariant *spherical polynomial* with $P_m(e) = 1$. It is well known (cf. [**26**, p. 448]) that

$$(P_m | P_m)_S = (\dim E_m)^{-1}. \tag{2.4}$$

THEOREM 2.1. *Let Z be an irreducible JB^*-triple of type (r, s, t). Let S denote the Shilov boundary of the open unit ball $D \subset Z$. Given any signature $m = (m_1,\ldots,m_r) \in \mathbf{N}_+^r$, the differential and integral scalar products are related for $p, q \in E_m$ as follows*:

$$\frac{(p|q)_Z}{(p|q)_S} = C_m = \prod_{j=1}^{r} \frac{(m_j + \frac{s}{2}(r-j) + t)!}{(\frac{s}{2}(r-j) + t)!}. \tag{2.5}$$

For the *proof* we assume, up to Lemma 2.6, that Z is a JB^*-algebra.

LEMMA 2.2. *Suppose $m, n \in \mathbf{N}_+^{r-1} \subset \mathbf{N}_+^r$ and $(B_v^* E_n | E_m)_S \neq 0$ for some $v \in Z$. Then*

$$C_{(m_1+k+1,\ldots,m_{r-1}+k+1,k+1)} = (k+1) C_{(n_1+k,\ldots,n_{r-1}+k,k)}$$

for all $k \in \mathbf{N}$.

PROOF. Given $q \in E_n \subset \mathcal{K}(Z)$, Proposition 1.5 and Lemma 1.9 imply

$$CN^{k+1}(B_v^* q) = (k+1) C_{(n_1+k,\ldots,n_{r-1}+k,k)} N^{k+1}(B_v^* q).$$

Since $(B_v^* q | E_m)_S \neq 0$ for some q, the assertion follows. Q.E.D.

LEMMA 2.3. *Suppose $m \in \mathbf{N}_+^{r-1}$. Then $(B_v^* E_n | E_m)_S \neq 0$ for $n := (m_1 + 1,\ldots, m_{r-1} + 1, 0)$ and $v := e_1$.*

PROOF. Put $l := v^*$. Then N_{r-1} and $\partial_l N$ coincide on Z_{r-1}. Further, $Av = 0$ for all vector fields A in the maximal nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{k}^{\mathbf{C}}$ spanned by all positive compact root spaces. This is clear if $A \in \mathfrak{k}_{pq}^{\mathbf{C}}$ and $1 \leq p < q \leq r$ (cf. [**27**, (1.8)]), and it follows as in the proof of [**27**, Lemma 3.4] if $A \in \mathfrak{k}_\alpha^{\mathbf{C}}$ and $\alpha | \mathfrak{h}_{-1} = 0$; now apply [**27**, Theorem 1.7]. Therefore $\partial_l N$ is $\mathfrak{n}_+$-invariant and hence can be written as a polynomial in $N_1,\ldots,N_{r-1}$ [**27**, Theorem 3.1]. It follows that $\partial_l N = N_{r-1}$ on Z. Since $N_n = N_m \cdot N_{r-1}$ by construction, (1.11) implies

$$(B_v^* N_n | N_m)_S = (N_n | \partial_l N \cdot N_m)_S = (N_n | N_n)_S > 0. \quad \text{Q.E.D.}$$

COROLLARY 2.4. *Suppose $n \in \mathbf{N}_+^{r-1}$ and $n_{r-1} > 0$. Then*

$$C_{(n_1+k,\ldots,n_{r-1}+k,k+1)} = (k+1) C_{(n_1+k,\ldots,n_{r-1}+k,k)}$$

for all $k \in \mathbf{N}$. Hence (2.5) *is true for all signatures in $\mathbf{N}_+^r$ if it holds for all signatures in $\mathbf{N}_+^{r-1}$.*

Let $B(a, b) := \Gamma(a)\Gamma(b)/\Gamma(a + b)$ denote the Beta-function.

LEMMA 2.5. *Denote by* N_m *and* P_m *the conical and spherical polynomials, resp., associated with* $m \in \mathbf{N}_+^r$. *Then*

$$\frac{(P_m|P_m)_S}{(N_m|N_m)_S} = \prod_{1\leqslant p<q\leqslant r} \frac{B\big(m_p - m_q, \frac{s}{2}(q+1-p)\big)}{B\big(m_p - m_q, \frac{s}{2}(q-p)\big)}.$$

PROOF. By [25] the K-module E_m has the highest weight

$$\mu = -\sum_{k=1}^{r} m_k \gamma_k,$$

where $\gamma_1, \ldots, \gamma_r$ denote the Harish-Chandra strongly orthogonal noncompact roots of $\mathfrak{k}^{\mathbf{C}}$ with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbf{C}}$ containing the vector fields $\{e_k e_k^* z\}(\partial/\partial z)$ for $1 \leqslant k \leqslant r$. Let $\mathfrak{h}^{\mathbf{C}} = \mathfrak{h}_1^{\mathbf{C}} \oplus \mathfrak{h}_{-1}^{\mathbf{C}}$ be the canonical decomposition with respect to $e \in S$ (cf. [27, Lemma 1.2]). Then the nonzero restrictions $\lambda = \alpha\,|\,\mathfrak{h}_{-1}^{\mathbf{C}}$ of positive compact roots α have the form $\lambda = (\gamma_q - \gamma_p)/2$ for $1 \leqslant p < q \leqslant r$, occurring with common multiplicity $m(\lambda) = s$ (cf. [25]). It follows that

$$\rho := \frac{1}{2}\sum_{\lambda>0} m(\lambda)\lambda = \frac{s}{2}\sum_{k=1}^{r}\left(k - \frac{r+1}{2}\right)\gamma_k.$$

Choose a K-invariant hermitian scalar product $(\ |\ )$ on $\mathfrak{k}^{\mathbf{C}}$ such that $iz(\partial/\partial z)$ is orthogonal to the derived ideal $\mathfrak{k}'$ and $(\gamma_p|\gamma_q) = 2\delta_{pq}$ for $1 \leqslant p, q \leqslant r$. The dual group of the compact semisimple Lie group K' generated by $\mathfrak{k}'$ has an Iwasawa decomposition. Therefore the quotient $(P_m|P_m)_S/(N_m|N_m)_S$ can be expressed as an integral (cf. [29, p. 211, proof of Theorem 3.3.1.1]). Evaluating this integral as in [29, 9.1.6, p. 325] yields

$$(P_m|P_m)_S/(N_m|N_m)_S = I(\mu+\rho)/I(\rho),$$

where for any linear form β on $\mathfrak{h}_{-1}^{\mathbf{C}}$ we put

$$I(\beta) := \prod_{\lambda>0} B\left(\frac{s}{2}, \frac{(\beta|\lambda)}{(\lambda|\lambda)}\right).$$

Now the result follows by direct computation. Q.E.D.

LEMMA 2.6. *Suppose* $m \in \mathbf{N}_+^r$. *Then*

$$\text{(2.6)} \quad \dim E_m = \prod_{1\leqslant p<q\leqslant r} \frac{m_p - m_q + \frac{s}{2}(q-p)}{\frac{s}{2}(q-p)} \frac{B\big(m_p - m_q, \frac{s}{2}(q-p-1)+1\big)}{B\big(m_p - m_q, \frac{s}{2}(q+1-p)\big)}$$

and

$$\text{(2.7)} \qquad \frac{1}{(N_m|N_m)_S} = \prod_{1\leqslant p<q\leqslant r} \frac{B\big(m_p - m_q, \frac{s}{2}(q-p-1)+1\big)}{B\big(m_p - m_q, \frac{s}{2}(q-p)+1\big)}.$$

PROOF. For the special cases $r \leqslant 2$ or $s = 1$, (2.6) follows easily by inspection and from Weyl's dimension formula [29, Theorem 2.4.1.6]. Hence we may assume $r \geqslant 3$ and $s \geqslant 2$. Then s is even and the positive roots α not vanishing on $\mathfrak{h}_{-1}^{\mathbf{C}}$ have the form

$$\alpha = (\gamma_q - \gamma_p)/2 \pm \beta_{pq}^{j},$$

where $1 \leqslant p < q \leqslant r$, $1 \leqslant j \leqslant s/2$ and β^j_{pq} are linear forms on $\mathfrak{h}^{\mathbf{C}}$ vanishing on $\mathfrak{h}^{\mathbf{C}}_{-1}$ (cf. [**25**]). Defining ρ_0 as in [**25**], it follows from [**25**, Lemma 12] that

$$\left\{\pm\left(\rho_0 \mid \beta^j_{pq}\right): 1 \leqslant j \leqslant s/2\right\} = \{\pm(j-1): 1 \leqslant j \leqslant s/2\}.$$

Another application of Weyl's formula gives (2.6). Combining (2.6), Lemma 2.5 and (2.4) yields (2.7). Q.E.D.

Proof of Theorem 2.1. We first prove Theorem 2.1 for irreducible *JB^*-algebras* Z by induction on the rank r of Z. If $r = 1$, then $Z = \mathbf{C}$ and (2.5) is clear. If $r = 2$, then $Z \approx \mathbf{C}^n$ is a "spin factor" of type $(2, n-2, 0)$. In this case the assertion (2.5) for signatures $(m_1, 0)$ follows from straightforward calculations (cf. [**2**]). By Lemma 2.4, (2.5) is true for all signatures $m \in \mathbf{N}^+_2$.

Now assume Z is of type $(r, s, 0)$ with $r \geqslant 3$. Then $\tilde{Z} := Z_{r-1}$, defined in (2.1), is an irreducible JB^*-algebra of type $(r-1, s, 0)$. By Lemma 2.4 it suffices to prove (2.5) for signatures $m = (m_1, \ldots, m_{r-1}, 0) \in \mathbf{N}^{r-1}_+$. Then, by construction, $N_m \in \mathcal{P}(\tilde{Z}) \subset \mathcal{P}(Z)$. Since the generic norms of Z and $\tilde{Z}$ agree on $\tilde{Z}$, it follows that $(N_m \mid N_m)_Z = (N_m \mid N_m)_{\tilde{Z}}$. Since the irreducible $\operatorname{Aut}(\tilde{Z})^0$-module generated by N_m has signature $(m_1, \ldots, m_{r-1})$, we can apply the induction hypothesis for $\tilde{Z}$ and (2.7) to conclude that (2.5) holds for Z.

Now assume Z is an irreducible *JB^*-triple* of type (r, s, t). Then $\tilde{Z} := Z_r$, defined in (2.1), is an irreducible JB^*-algebra of type $(r, s, 0)$. Further, the group L (cf. (2.3)) commutes with the Peirce projection onto $\tilde{Z}$. By (2.3) the spherical polynomial P_m associated with the signature $m \in \mathbf{N}^r_+$ lies in $\mathcal{P}(\tilde{Z})$, whence

$$(P_m \mid P_m)_Z = (P_m \mid P_m)_{\tilde{Z}}.$$

According to (2.4) the integral scalar products are related to the dimensions of the corresponding K-module $E_m \subset \mathcal{P}(Z)$ and $\operatorname{Aut}(\tilde{Z})^0$-module $\tilde{E}_m \subset \mathcal{P}(\tilde{Z})$. Therefore the assertion follows from the first part of the proof, applied to $\tilde{Z}$, and from the following lemma. Q.E.D.

Lemma 2.7. *Suppose Z is an irreducible JB^*-triple of type (r, s, t). Then*

$$\frac{\dim E_m}{\dim \tilde{E}_m} = \prod_{j=1}^{r} \frac{B\left(m_j, \frac{s}{2}(r-j)+1\right)}{B\left(m_j, \frac{s}{2}(r-j)+t+1\right)}.$$

Proof. Denote by $F_{(n_1, \ldots, n_k)}$ the irreducible $\mathrm{GL}(k, \mathbf{C})$-representation with signature $n_1 \geqslant \cdots \geqslant n_k \geqslant 0$. The classical irreducible JB^*-triples Z with $t > 0$ are of type $(r, 2, t)$ or $(r, 4, 2)$. In these cases we have, respectively,

$$E_m \approx F_{(m_1, \ldots, m_r)} \otimes F_{(m_1, \ldots, m_r, 0, \ldots, 0)} \quad (t \text{ zeros}),$$

$$\tilde{E}_m \approx F_{(m_1, \ldots, m_r)} \otimes F_{(m_1, \ldots, m_r)}, \quad \text{or}$$

$$E_m \approx F_{(m_1, m_1, \ldots, m_r, m_r, 0)}, \qquad \tilde{E}_m \approx F_{(m_1, m_1, \ldots, m_r, m_r)}.$$

In both cases Lemma 2.7 follows from [**12**, (1.2.10)]. For the exceptional domain of type $(2, 6, 4)$, apply Weyl's dimension formula to $K \approx \mathrm{Spin}(10) \cdot \mathbf{T}$. Q.E.D.

We now apply Theorem 2.1 to analyze the fine structure of certain Toeplitz operators.

LEMMA 2.8. *Suppose Z is an irreducible JB^*-algebra, $v \in Z$ and $m \in \mathbf{N}_+^{r-1}$. Then $B_v(E_m) \subset E_n$, where $n := (m_1 + 1, \ldots, m_{r-1} + 1, 0)$.*

PROOF. Suppose $n \in \mathbf{N}_+^{r-1}$ satisfies $(B_v(E_m) | E_n)_S \neq 0$. By Lemma 2.4 it follows that

$$C_{(m_1+k+1,\ldots,m_{r-1}+k+1,k+1)} = (k+1)C_{(n_1+k,\ldots,n_{r-1}+k,k)}$$

for all $k \in \mathbf{N}$. By Theorem 2.1 (applied to k and $k+1$), this implies

$$\prod_{j=1}^{r} \left(m_j + k + 1 + \tfrac{a}{2}(r-j)\right) = \prod_{j=1}^{r} \left(n_j + k + \tfrac{a}{2}(r-j)\right),$$

whence $m_j + 1 = n_j$ for all $j < r$. Q.E.D.

Given a signature $m = (m_1, \ldots, m_r) \in \mathbf{N}_+^r$, define, for $1 \leqslant j \leqslant r$,

$$m \pm \varepsilon_j := (m_1, \ldots, m_{j-1}, m_j \pm 1, m_{j+1}, \ldots, m_r)$$

and put $E_{m \pm \varepsilon_j} := \{0\}$ if $m \pm \varepsilon_j$ is not a signature.

LEMMA 2.9. *Suppose Z is an irreducible JB^*-triple and $m \in \mathbf{N}_+^r$. Then the conical polynomial N_m satisfies*

$$\left(v\frac{\partial}{\partial z}\right)N_m \in \sum_{k<j\leqslant r}^{\oplus} E_{m-\varepsilon_j}$$

for all $v \in \sum_{0\leqslant p\leqslant q\leqslant r-k}^{\oplus} Z_{pq}$ and $0 \leqslant k < r$.

PROOF. We may assume $v \in Z_{pq}$ for $0 \leqslant p \leqslant q \leqslant r-k$. Further, we may assume Z is a JB^*-algebra since $v(\partial/\partial z)N_m = 0$ if $p = 0$. The proof is by induction on r, the case $r = 1$ being trivial. If $r \geqslant 2$, then Z_{r-1}, defined in (2.1), is an irreducible JB^*-algebra of rank $r-1$. Now $N_m = N_n \cdot N^{m_r}$, where $n := (m_1 - m_r, \ldots, m_{r-1} - m_r, 0)$. For $l := v^*$, the product rule implies

$$\left(v\frac{\partial}{\partial z}\right)N_m = N^{m_r}\left(v\frac{\partial}{\partial z}N_n\right) + m_r N_n N^{m_r-1}\partial_l N.$$

To treat the first summand, observe that $(v(\partial/\partial z))N_n = 0$ if $p = 1$. Hence we may assume $0 \neq v \in Z_{r-1}$ and, therefore, $0 \leqslant k < r-1$. The induction hypothesis applied to Z_{r-1} yields

$$\left(v\frac{\partial}{\partial z}\right)N_n \in \sum_{k<j<r}^{\oplus} E_{n-\varepsilon_j}.$$

To treat the second summand, we may assume $m_r = 1$. If $p < q \leqslant r-k$, then $A := \{ve_q^*z\}(\partial/\partial z) \in \mathfrak{n}_+$ by [27, Theorem 1.7] and a straightforward computation shows

$$w := \exp(A)(e_q) = e_q + v/2 + ae_p$$

for some $a \in \mathbf{C}$. Since N is $\mathfrak{n}_+$-invariant, it follows that

$$\left(w\frac{\partial}{\partial z}N\right)(\exp(A)z) = \left(e_q\frac{\partial}{\partial z}N\right)(z).$$

Hence we may assume $p = q$ and $v = e_q$ for $1 \leqslant q \leqslant r - k$. Since $N_n \in \mathcal{H}(Z)$, it follows that $\partial_l N \cdot N_n = B_v(N_n) + N \cdot A_v(N_n)$. By Lemma 2.8 $B_v(N_n) \in E_{(m_1,\ldots,m_{r-1},0)}$. On the other hand, if $q \geqslant 2$, the induction hypothesis, applied to $v \in Z_{r-1}$, together with Proposition 1.8 and Corollary 1.4 imply

$$A_v(N_n) \in \sum_{k<j<r}^{\oplus} E_{n-\varepsilon_j}.$$

If $q = 1$, then $\partial_l N = N_{r-1}$ implies $A_v(N_n) = 0$. Q.E.D.

In view of Corollary 1.4 and Proposition 1.5, Lemma 2.9 has the following consequence:

COROLLARY 2.10. *Suppose $v \in Z$. Then*

$$\left(v\frac{\partial}{\partial z}\right)E_m \subset \sum_{1\leqslant j\leqslant r}^{\oplus} E_{m-\varepsilon_j}$$

and

$$\{zv^*z\}\frac{\partial}{\partial z}E_m \subset \sum_{1\leqslant j\leqslant r}^{\oplus} E_{m+\varepsilon_j}.$$

Combining Corollary 1.4, Proposition 1.5, Theorem 2.1 and Corollary 2.10 enables us to describe the relationship between "basic" Toeplitz operators and differential operators in an explicit form:

THEOREM 2.11. *Suppose Z is an irreducible JB^*-triple of type (r, s, t). For $v \in Z$ put $l := v^*$. Then*

$$T_l^* p = \sum_{j=1}^{r}\left(m_j + \frac{s}{2}(r-j) + t\right)^{-1}\left(v\frac{\partial}{\partial z}p\right)_{m-\varepsilon_j}$$

and

$$T_l p = \sum_{j=1}^{r}\left(m_j - \frac{s}{2}(j-1)\right)^{-1}\left(\{zv^*z\}\frac{\partial}{\partial z}p\right)_{m+\varepsilon_j}$$

for all $p \in E_m$ and $m \in \mathbf{N}_+^r$, the subscript $m \pm \varepsilon_j$ denoting the respective Peter-Weyl component.

COROLLARY 2.12. *The commutator $[T_l^*, T_l]$ is a "diagonal" operator, i.e. $[T_l^*, T_l]E_m \subset E_m$ for all signatures $m \in \mathbf{N}_+^r$.*

PROOF. Suppose $1 \leqslant j, k \leqslant r$ and $j \neq k$. Then

$$v\frac{\partial}{\partial z}(T_l p)_{m+\varepsilon_k} - T_l\left(v\frac{\partial}{\partial z}p\right)_{m-\varepsilon_j}$$

has vanishing $(m + \varepsilon_k - \varepsilon_j)$-component, if $m \in \mathbf{N}_+^r$ and $p \in E_m$, since $[v(\partial/\partial z), T_l] = l(v) \cdot \mathrm{id}$ is a diagonal operator. Now apply Theorem 2.11. Q.E.D.

3. The Toeplitz C^*-algebra. The mapping $f \mapsto T_f$ from $L^\infty(S)$ into the C^*-algebra $\mathcal{L}(H)$ of all bounded linear operators on the Hilbert space $H := H^2(S)$ is by definition (completely) positive but not a homomorphism. In particular, Toeplitz operators are in general not normal and do not necessarily commute. The appropriate object to study Toeplitz operators is therefore the C^*-algebra $C^*(T_f\colon f \in \Sigma)$ generated by all Toeplitz operators T_f with f belonging to a certain "symbol algebra" $\Sigma \subset L^\infty(S)$. For $\Sigma = L^\infty(S)$, the structure of the corresponding C^*-algebra is unknown even in the simplest case of the unit circle $S \subset \mathbf{C}$ (cf. [**9**, §7]). For *continuous* symbols $f \in \mathcal{C}(S)$ however, there exists a satisfying structure theory for the *Toeplitz C^*-algebra* $\mathfrak{T} := C^*(T_f\colon f \in \mathcal{C}(S))$ associated with the Shilov boundary S of a bounded symmetric domain D of arbitrary rank. The following results concerning the structure of $\mathfrak{T}$ generalize results of C. Berger, L. Coburn and A. Korányi [**1**, **2**, **7**] for domains of rank $\leqslant 2$.

LEMMA 3.1. *The Toeplitz C^*-algebra $\mathfrak{T}$ acts irreducibly on $H^2(S)$.*

PROOF. Put $H := H^2(S)$ and let $\pi \in \mathcal{L}(H)$ be a projection commuting with $\mathfrak{T}$. Define $f := \pi(1) \in H$. Then $\pi(g) = \pi(T_g 1) = T_g f = gf$ for all $g \in \mathcal{P}(Z)$. Given $z \in D$, define $\mathcal{S}_z \in H$ by $\mathcal{S}_z(v) := \mathcal{S}(v, z)$ for all $v \in S$, $\mathcal{S}$ denoting the Szegö kernel (cf. [**17**, p. 344]). It follows that

$$\begin{aligned}(\pi(\mathcal{S}_z)\,|\,g)_S &= (\mathcal{S}_z\,|\,\pi(g))_S = (\mathcal{S}_z\,|\,gf)_S = g(z)f(z)\\ &= f(z)(\mathcal{S}_z\,|\,g)_S\end{aligned}$$

using the reproducing property of $\mathcal{S}$. Since $g \in \mathcal{P}(Z)$ is arbitrary, $\overline{f(z)}$ is an eigenvalue of π, hence $f = 0$ or $f = 1$ since D is a domain. Q.E.D.

For irreducible domains of tube type there exists a tensor product representation (cf. §1)

$$H^2(S) = \overline{\mathcal{H}(Z)} \otimes H^2(\mathbf{T}). \tag{3.1}$$

Let $\mathcal{C}$ denote the unital C^*-algebra (acting on $\overline{\mathcal{H}(Z)}$) generated by all operators A_v for $v \in Z$. Then also $B_v \in \mathcal{C}$ and for fixed $t \in \mathbf{T}$, the operators $A_v + tB_v$ for $v \in Z$ generate an abelian C^*-subalgebra of $\mathcal{C}$, as follows from Proposition 1.8. In particular, $\mathcal{C}$ is generated by two abelian C^*-subalgebras (cf. [**23**]).

PROPOSITION 3.2. *Let Z be an irreducible JB^*-algebra. Then $\mathfrak{T}$ is a C^*-subalgebra of the C^*-algebra tensor product $\mathcal{C} \otimes \mathfrak{T}_{\mathbf{T}}$ (with respect to* (3.1)) *and the closed commutator ideal $\mathfrak{T}'$ of $\mathfrak{T}$ satisfies $\mathfrak{T}' = \mathcal{C} \otimes \mathcal{K}_{\mathbf{T}}$. Here $\mathfrak{T}_{\mathbf{T}}$ (resp. $\mathcal{K}_{\mathbf{T}}$) denotes the Toeplitz C^*-algebra (resp. the C^*-algebra of all compact operators) on $H^2(\mathbf{T})$.*

PROOF. The first statement is immediate from Proposition 1.7. Denote by ξ the identity function on $\mathbf{T}$. Then $P := [T_\xi^*, T_\xi] = \mathrm{id} - T_\xi T_\xi^*$ is the orthogonal projection from $H^2(\mathbf{T})$ onto $\mathbf{C} \cdot 1$, hence $P \in \mathcal{K}_{\mathbf{T}}$. By Proposition 1.7 the operators $\mathrm{id} \otimes P = [T_N^*, T_N]$ and $A_v^* \otimes P = [T_N^*, T_N]T_l$ belong to $\mathfrak{T}'$, where $v \in Z$ and $l := v^*$. Hence $\mathcal{C} \otimes \mathcal{K}_{\mathbf{T}} \subset \mathfrak{T}'$ since $\mathcal{K}_{\mathbf{T}}$ is a simple C^*-algebra. Conversely, the commutation rules (Proposition 1.8) imply $[T_l^*, T_l] = B_v^* B_v \otimes P \in \mathcal{C} \otimes \mathcal{K}_{\mathbf{T}}$. Since $\mathfrak{T}'$ is generated (as an ideal) by such commutators, the assertion follows with Proposition 1.7. Q.E.D.

By [**20**, Theorem 6.3], the "boundary components" of a bounded symmetric domain D are related to the tripotents of the associated JB^*-triple Z. It will now be shown that to each tripotent $e \in Z$ there corresponds an *irreducible representation* of the Toeplitz C^*-algebra $\mathfrak{T}$. Let $Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e)$ be the Peirce decomposition induced by e [**20**, Theorem 3.13], where $Z_j(e) := \{z \in Z: \{ee^*z\} = jz\}$. Let S_e be the Shilov boundary of the bounded symmetric domain $D \cap Z_e$ contained in the JB^*-subtriple $Z_e := Z_0(e)$ of Z. Endow S_e with the probability measure μ_e invariant under $K_e := \mathrm{Aut}(Z_e)^0$. Then $e \oplus S_e \subset S$, since S consists of all tripotents of maximal rank [**20**, Theorem 6.5]. Given $f \in \mathcal{C}(S)$, define $f_e \in \mathcal{C}(S_e)$ by $f_e(w) := f(e + w)$ for all $w \in S_e$. Denoting the generic trace of Z by $(\ |\ )$ put $H_e(z) := \exp(z \mid e)$ and define "peaking functions"

$$h_e^i(z) := H_e^i(z)/\|H_e^i\|_S$$

for all $i \in \mathbf{N}$, $\|\ \|_S$ denoting the norm in $H^2(S)$. For sequences (f_i) and (g_i) in $H^2(S)$, we shall write $f_i \sim g_i$ if $\lim_{i \to \infty} \|f_i - g_i\|_S = 0$.

LEMMA 3.3. *Let Z be a JB^*-triple and $e \in Z$ a tripotent. Then for any $f \in \mathcal{C}(S)$:*

$$\mu_e(f_e) = \lim_{i \to \infty} \int_S f |h_e^i|^2 \, d\mu.$$

PROOF. Suppose $\sup|f_e(S_e)| \leqslant \delta < \varepsilon$. According to [**20**, Lemma 6.2], there exist neighborhoods $V \subset U \subset S$ of $e \oplus S_e$ such that $\sup|f(U)| \leqslant \varepsilon$ and

$$\sup_{z \notin U} |H_e(z)| < \inf_{z \in V} |H_e(z)|.$$

For the probability measures $\mu_i := |h_e^i|^2 \cdot \mu$ this implies $\lim_{i \to \infty} \mu_i(1_{S \setminus U} f) = 0$ and, hence, letting $\varepsilon \to \delta$,

$$\lim_{i \to \infty} \sup|\mu_i(f)| \leqslant \delta.$$

It follows that every accumulation point λ of $(\mu_i)_{i \in \mathbf{N}}$ actually defines a probability measure on S_e, which is obviously K_e-invariant since any $\gamma \in K_e$ has an extension $g \in K$ fixing e. By the uniqueness of μ_e, a compactness argument implies $\lim_{i \to \infty} \mu_i(f) = \mu_e(f_e)$. Q.E.D.

COROLLARY. 3.4. *For all $p \in \mathcal{P}(Z)$ and $q \in \mathcal{P}(Z_e) \subset \mathcal{P}(Z)$ (via the Peirce projection) we have $T_p(h_e^i q) \sim h_e^i(T_{p_e} q)$.*

PROOF. Since $p - p_e$ vanishes on $e \oplus S_e$, Lemma 3.3 yields the assertion. Q.E.D.

In order to prove a similar result for the adjoint operators, some preparation is needed. In the next two lemmas, Z is assumed to be irreducible. There exists a frame $\{e_1, \ldots, e_r\}$ of Z such that $e = e_{r+1-k} + \cdots + e_r$ for some $k \in \{0, \ldots, r\}$. For any fixed $\mu \in \mathbf{N}_+^{r-k}$, let P_μ denote the orthogonal projection of $H^2(S)$ onto the Hilbert sum of all K-modules E_m with signatures of the form $m = (m_1, \ldots, m_k, \mu)$.

LEMMA 3.5. *For $\mu = (m_{k+1}, \ldots, m_r) \in \mathbf{N}_+^{r-k}$ put $m := (m_{k+1}, \ldots, m_{k+1}, \mu) \in \mathbf{N}_+^r$. Then*

$$pN_m \in P_\mu(H^2(S)), \tag{3.2}$$

$$T_l^*(pN_m) = \sum_{k < j \leqslant r} \left(m_j + \frac{s}{2}(r - j) + t\right)^{-1} P_{\mu - \varepsilon_j}\left(p\left(v \frac{\partial}{\partial z} N_m\right)\right) \tag{3.3}$$

whenever $p \in \mathcal{P}(Z_1(e))$, $v \in Z_e$ and $l := v^*$.

PROOF. Every γ in the commutator group of $\mathrm{Aut}(Z_1(e))^0$ has an extension $g \in K$ satisfying $N_m \circ g = N_m$ and $g|Z_e = \mathrm{id}$. Hence we may assume $p = N_1^{l_1} \cdots N_k^{l_k}$ and, therefore, $pN_m = N_n$ for some signature $n = (n_1, \dots, n_k, \mu) \in \mathbf{N}_+^r$. Then (3.2) is clear. Since $v \in Z_e$, Lemma 2.9 and Theorem 2.11 imply

$$T_l^*(pN_m) = \sum_{k<j\leq r} \left(m_j + \frac{s}{2}(r-j) + t\right)^{-1} \left(v\frac{\partial}{\partial z}(pN_m)\right)_{n-\varepsilon_j}. \quad \text{Q.E.D.}$$

LEMMA 3.6. *Suppose $q \in \mathcal{P}(Z_e)$ and $\nu \in \mathbf{N}_+^{r-k}$. Then*

$$P_\nu(h_e^i q) \sim h_e^i q_\nu.$$

PROOF. We may assume q is the conical polynomial on Z_e with signature $\mu \in \mathbf{N}_+^{r-k}$. Since $N_m - q$ vanishes on $e \oplus S_e$, Lemma 3.3 implies $P_\nu(h_e^i q) \sim P_\nu(h_e^i N_m)$. If $\mu \neq \nu$, then $P_\nu(h_e^i N_m) = 0$ by (3.2) and $q_\nu = 0$. If $\mu = \nu$, then (3.2) implies

$$P_\nu(h_e^i N_m) = h_e^i N_m \sim h_e^i q = h_e^i q_\nu. \quad \text{Q.E.D.}$$

LEMMA 3.7. *Let Z be a JB*-triple, $p \in \mathcal{P}(Z)$ and $q \in \mathcal{P}(Z_e)$. Then $T_p^*(h_e^i q) \sim h_e^i(T_{p_e}^* q)$.*

PROOF. By simple tensor product considerations, we may assume Z is irreducible. Further, using multiplicative properties, we may assume $p = v^*$ is linear. If $v \in Z_e^\perp$, then $p - p_e$ vanishes on $e \oplus S_e$ and p_e is constant, hence applying the Szegö projection π and Lemma 3.3 yields the assertion. Now assume $v \in Z_e$. Since every $\gamma \in K_e$ has an extension $g \in K$ fixing e, we may assume q is the conical polynomial on Z_e with signature $\mu = (m_{k+1}, \dots, m_r) \in \mathbf{N}_+^{r-k}$. Since Z_e is of type $(r-k, s, t)$ and $r - j = (r-k) - (j-k)$ for $j > k$, Theorem 2.12 implies

$$T_{p_e}^* q = \sum_{k<j\leq r} \left(m_j + \frac{s}{2}(r-j) + t\right)^{-1} \left(v\frac{\partial}{\partial z}q\right)_{\mu-\varepsilon_j}. \tag{3.4}$$

Since $v(\partial/\partial z)(N_m - q)$ vanishes on $e \oplus S_e$, Lemmas 3.3 and 3.6 imply

$$P_{\mu-\varepsilon_j}\left(h_e^i\left(v\frac{\partial}{\partial z}N_m\right)\right) \sim P_{\mu-\varepsilon_j}\left(h_e^i\left(v\frac{\partial}{\partial z}q\right)\right) \sim h_e^i\left(v\frac{\partial}{\partial z}q\right)_{\mu-\varepsilon_j}.$$

Combining this with (3.3) and (3.4), we obtain

$$T_p^*(h_e^i q) \sim T_p^*(h_e^i N_m) \sim h_e^i(T_{p_e}^* q). \quad \text{Q.E.D.}$$

Denote by $\mathcal{T}_0$ the $*$-algebra generated by all Toeplitz operators T_p for $p \in \mathcal{P}(Z)$.

THEOREM 3.8. *For each tripotent e of a JB*-triple Z there exists an irreducible representation ("e-symbol") σ_e of the Toeplitz C*-algebra $\mathcal{T}$ on the Hardy space $H^2(S_e)$, uniquely determined by one of the following properties:*

(i) $\sigma_e(T_f) = T_{f_e}$ *for all* $f \in \mathcal{C}(S)$;

(ii) $\lim_{i\to\infty}\|A(h_e^i q) - h_e^i(\sigma_e(A)q)\|_S = 0$, *if* $A \in \mathcal{T}_0$ *and* $q \in \mathcal{P}(Z_e) \subset \mathcal{P}(Z)$.

PROOF. Let $\mathcal{Q}$ denote the set of all operators $A \in \mathcal{T}_0$ such that (ii) is true for some operator $\sigma_e(A)$ on $\mathcal{P}(Z_e)$. By Lemma 3.3, $\sigma_e(A)$ is uniquely determined by A and

$$\|\sigma_e(A)\| \leq \|A\| \tag{3.5}$$

for the respective operator norms. By definition $\mathcal{Q}$ is an algebra and σ_e: $\mathcal{Q} \to \mathcal{L}(H^2(S_e))$ is a homomorphism. For every $p \in \mathcal{P}(Z)$, it follows from Corollary 3.4 that $T_p \in \mathcal{Q}$ and $\sigma_e(T_p) = T_{p_e}$. Further, Lemma 3.7 implies $T_p^* \in \mathcal{Q}$ and $\sigma_e(T_p^*) = T_{p_e}^*$. Hence $\mathcal{Q} = \mathfrak{T}_0$ and, by (3.5), σ_e has a unique extension to a C^*-algebra homomorphism on $\mathfrak{T}$ satisfying (i). Since $\sigma_e(\mathfrak{T})$ is the Toeplitz C^*-algebra on S_e, σ_e is irreducible by Lemma 3.1. Q.E.D.

PROPOSITION 3.9. *The representations σ_e of $\mathfrak{T}$ for tripotents $e \in Z$ are mutually inequivalent.*

PROOF. Suppose e and c are different tripotents of rank k and j, respectively. Then $e \oplus S_e \neq c \oplus S_c$. This is clear if $k \neq j$ and it follows from [**20**, Lemma 6.2] if $k = j$. By Urysohn's theorem there exists $f \in \mathcal{C}(S)$ vanishing on $e \oplus S_e$ but not on $c \oplus S_c$. Hence $T_f \in \ker(\sigma_e) \setminus \ker(\sigma_c)$. Q.E.D.

The irreducible representations corresponding to maximal tripotents $e \in S$ are 1-dimensional, i.e. characters. We will now show that these are the only characters of $\mathfrak{T}$.

LEMMA 3.10. *Let $\mathfrak{T}'$ denote the closed commutator ideal of $\mathfrak{T}$. Then $T_f T_g - T_{fg} \in \mathfrak{T}'$ whenever $f, g \in \mathcal{C}(S)$.*

PROOF. By Stone-Weierstrass the linear space generated by all functions $\bar{p}q$ for $p, q \in \mathcal{P}(Z)$ is a dense $*$-subalgebra of $\mathcal{C}(S)$. Hence the assertion follows from

$$T_{\bar{p}q} T_{\bar{f}g} - T_{\bar{p}q\bar{f}g} = T_p^* \left[T_q, T_f^*\right] T_g \in \mathfrak{T}'. \quad \text{Q.E.D.}$$

PROPOSITION 3.11. *The mapping $f \mapsto T_f$ induces a C^*-algebra isomorphism $\mathcal{C}(S) \approx \mathfrak{T}/\mathfrak{T}'$. Hence every operator $A \in \mathfrak{T}$ has a unique representation $A = T_f + A'$, where $f \in \mathcal{C}(S)$ and $A' \in \mathfrak{T}'$.*

PROOF. By Lemma 3.10 the mapping $f \mapsto T_f$ is a homomorphism onto the C^*-algebra $\mathfrak{T}/\mathfrak{T}'$. Since $H^2(S_e) \approx \mathbf{C}$ for all $e \in S$, the homomorphisms σ_e for all $e \in S$ yield a homomorphism $\mathfrak{T} \to \mathcal{C}(S)$ vanishing on $\mathfrak{T}'$. Obviously these homomorphisms are reciprocal. Q.E.D.

Generalizing Proposition 3.11, we can show that the representations σ_e induced by tripotents $e \in Z$ constitute *all* irreducible representations of $\mathfrak{T}$. This implies, by general C^*-algebra theory [**8**, Chapter 10], that $\mathfrak{T}$ is a *solvable* C^*-algebra of length $r := \text{rank}(D)$ (cf. [**10**]) with *spectral components* $S_k := \{e \in Z\text{: } e \text{ tripotent of rank } k\}$ for $0 \leqslant k \leqslant r$. The key fact is that an operator $A \in \mathfrak{T}$ satisfying $\sigma_e(A) = 0$ for all nonzero tripotents e is actually *compact*. The proof of this is based on the explicit construction of completely positive "smooth" cross-sections τ_k of the k-symbol homomorphisms $\sigma_k := (\sigma_e)_{e \in S_k}$ for $0 \leqslant k \leqslant r$, generalizing the Toeplitz map $f \mapsto T_f$ (for $k = r$) and also the "Friedrichs map" for pseudo-differential operators on spheres. The details will appear in a forthcoming publication.

REFERENCES

1. C. A. Berger and L. A. Coburn, *Wiener-Hopf operators on U_2*, Integral Equations Operator Theory **2** (1979), 139–173.

2. C. A. Berger, L. A. Coburn and A. Korányi, *Opérateurs de Wiener-Hopf sur les sphères de Lie*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), 989–991.

3. L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Ann. of Math. Studies, no. 99, Princeton Univ. Press, Princeton, N. J., 1981.

4. L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Invent. Math. **50** (1979), 249–272.

5. H. Braun and M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin and New York, 1966.

6. R. Braun, W. Kaup and H. Upmeier, *A holomorphic characterization of Jordan C^*-algebras*, Math. Z. **161** (1978), 277–290.

7. L. A. Coburn, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math. J. **23** (1973), 433–439.

8. J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1977.

9. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

10. A. Dynin, *Inversion problem for singular integral operators: C^*-approach*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 4668–4670.

11. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

12. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Transl. Math. Monos., vol. 6, Amer. Math. Soc., Providence, R.I., 1963.

13. J. Janas, *Toeplitz operators related to certain domains in* $\mathbf{C}^n$, Studia Math. **54** (1975), 73–79.

14. N. P. Jewell and S. G. Krantz, *Toeplitz operators and related function algebras on certain pseudoconvex domains*, Trans. Amer. Math. Soc. **252** (1979), 297–311.

15. K. D. Johnson, *On a ring of invariant polynomials on a hermitian symmetric space*, J. Algebra **67** (1980), 72–80.

16. M. Koecher, *An elementary approach to bounded symmetric domains*, Rice Univ., Houston, 1971.

17. A. Koráanyi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. of Math. (2) **82** (1965), 332–350.

18. D. A. Levine, *Systems of singular integral operators on spheres*, Trans. Amer. Math. Soc. **144** (1969), 493–522.

19. O. Loos, *Jordan pairs*, Springer-Verlag, Berlin and New York, 1975.

20. ______, *Bounded symmetric domains and Jordan pairs*, Univ. of California, Irvine, 1977.

21. P. S. Muhly and J. N. Renault, *C^*-algebras of multi-variable Wiener-Hopf operators*, Trans. Amer. Math. Soc. **274** (1983), 1–44.

22. R. Narasimhan, *Several complex variables*, Univ. of Chicago, Chicago, 1971.

23. S. C. Power, *Commutator ideals and pseudo-differential C^*-algebras*, Quart. J. Math. Oxford Ser. (2) **31** (1980), 467–489.

24. I. Raeburn, *On Toeplitz operators associated with strongly pseudo-convex domains*, Studia Math. **63** (1979), 253–258.

25. W. Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Invent. Math. **9** (1969), 61–80.

26. M. Takeuchi, *Polynomial representations associated with symmetric bounded domains*, Osaka J. Math. **10** (1973), 441–475.

27. H. Upmeier, *Jordan algebras and harmonic analysis on symmetric spaces*, (preprint).

28. U. Venugopalkrishna, *Fredholm operators associated with strongly pseudoconvex domains*, J. Funct. Anal. **9** (1972), 349–373.

29. G. Warner, *Harmonic analysis on semi-simple Lie groups*. I, II, Springer-Verlag, Berlin and New York, 1972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104